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Canonical variables and analysis on so$(n, 2)$†

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Abstract

The approach of Berezin to the quantization of so(n,2) via generalized coherent states is considered in detail. A family of n commuting observables is found in which the basis for an associated Fock-type representation space is expressed. An interesting feature is that computations can be done by explicit matrix calculations in a particular basis. The basic technical tool is the Leibniz function, the inner product of coherent states.

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1. Introduction

In the papers [6–8], Berezin presents an approach to quantization using generalized coherent states, as explained by Perelomov [22], also see [3]. The surveys of Ali *et al* [2] and of Klauder and Skagerstam [16] provide background as well as recent developments in these areas.

In this paper, we show how these ideas can be adapted to a method for explicit calculations. In particular, we work out details for the pseudo-orthogonal group so(n,2). A matrix formulation provides for basic computations and for finding the Leibniz function (kernel function). The observables—natural variables for analysis on so(n,2)—are found and their joint spectral density is shown to be the Wishart distribution on the Minkowski cone. Interestingly, we find the generating function for a polynomial basis of the representation space related to classical orthogonal polynomials. We conclude by showing how the Lie algebra is recovered from the Leibniz function.

Work most closely related to this paper is that of Onofri [21] and Berceanu and Gheorghe [5]. The coherent state methods given by Hecht [14] are closely related to the present article as well. A principal ingredient of our method is relating the Cartan decomposition (and generalizations) of the Lie algebra to Berezin's quantization, as explained in section 2. A novel contribution is the algebraic/analytic technique of deriving directly from the Leibniz function a realization of the Lie algebra in terms of combinatorial raising and lowering operators (bosons). Note also that our approach is effectively the Fourier image of that in the literature, such as [14].

† A talk based on an earlier version of this work was presented by PF and MG in the special session on Algebraic Methods in Statistics organized by G Letac at the 1997 AMS meeting in Montreal. This paper is based on the PhD dissertation of MG [1].

We feel that ours has certain advantages, such as identification of the Bessel operator in section 4 as well as for operator calculations.

In addition to the works cited above, in the mathematics literature we have found the book by Hua [15] and the paper of Wolf [23] very useful. An exposition of the present authors' theory with emphasis on connections with probability is given in [12]. A major aspect of the mathematical point of view is the theory of symmetric cones and Jordan algebras. See [11] for analysis in that context.

The significance of the pseudo-Euclidean group SO(n,2) in physics is well-known. It serves as a 'linearization' of the conformal group of Minkowski space $\boldsymbol{R}^{n-1,1}$ (see e.g., [13]), the symmetry group of Maxwell's equations. Also, the group SO(n,2) plays an important rôle in the n-dimensional Kepler problem, where the compactified phase space (the Moser phase space) coincides with a coadjoint orbit of the dynamical group SO(n+1,2) [10, 17, 20]. In another context, the group SO(4,2) serves as the spectrum-generating symmetry group of the hydrogen atom [4, 18].

2. Cartan decomposition and Berezin theory

Consider a Lie algebra $\mathfrak{g}$. At the heart of our construction is the existence of two abelian subalgebras $\mathcal{R}$ and $\mathcal{L}$ of the same dimension n, such that the Lie algebra they generate is $\mathfrak{g}$ itself: $\mathfrak{g} = \text{gen}\,\{\mathcal{L}, \mathcal{R}\}$.

An important case of such a structure is the *Cartan decomposition* for symmetric Lie algebras, where $\mathfrak{g}$ has the form

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{R} \tag{1}$$

with $\mathcal{L}$ and $\mathcal{R}$ satisfying $[\mathcal{L}, \mathcal{R}] \subseteq \mathcal{K}$, $[\mathcal{K}, \mathcal{R}] \subseteq \mathcal{R}$, and $[\mathcal{K}, \mathcal{L}] \subseteq \mathcal{L}$.

Denote bases for $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{K}$ by $\{R_j\}$, $\{L_j\}$, and $\{\rho_A\}_{1\leqslant A\leqslant m}$, respectively.

Remark 2.1. Later in the paper we will give a Cartan decomposition of so(n,2). The ρ elements will be taken as generators of rotations in the purely spatial or temporal sectors of $\boldsymbol{R}^{n,2}$, while the $\mathcal{L}$ and $\mathcal{R}$ elements will be certain combinations of boosts.

A typical element $X \in \mathfrak{g}$ is of the form

$$X = v'_j R_j + u'_A \rho_A + w'_j L_j \tag{2}$$

for some $(2n+m)$-tuple (v', u', w'). We can express exponentiation of X to an element of the group either by the standard exponential map, or via factorization into subgroups corresponding to the decomposition of the Lie algebra, thus

$$\mathrm{e}^X = \exp(v_i R_i) \left(\prod_A \exp(u_{\dot{A}} \rho_{\dot{A}}) \right) \exp(w_j L_j). \tag{3}$$

(We use the convention of summation over repeated indices, unless they are dotted; there is no summation over $\dot{A}$ above). Clearly, the coordinates (v, u, w) versus (v', u', w') are mutually dependent as they represent in (3) the same group element.

Our general goal is to construct a representation space for the enveloping algebra of $\mathfrak{g}$ and then find an abelian subalgebra of self-adjoint operators to take as our observables of interest.

First, let us construct a Hilbert space $\mathcal{H}$ spanned by a basis

$$|k_1, k_2, \ldots, k_n\rangle = R_1^{k_1} \cdots R_n^{k_n} \Omega \tag{4}$$

where Ω is a vacuum state.

Define the action of the algebra elements on the vacuum state thus:

(i) $\hat{R}_j\Omega = R_j\Omega$

(ii) $\hat{L}_j\Omega = 0$

(iii) $\hat{\rho}_A\Omega = \tau_A\Omega$

where τ_A are constants. Next, assume that $\mathcal{H}$ admits a symmetric scalar product (not necessarily hermitian!) in some number field, such that the ladder operators are mutually adjoint with respect to it:

$$\hat{R}_i^* = \hat{L}_i.$$

Thus, there is a one-to-one map $\mathcal{R} \leftrightarrow \mathcal{L}$ that admits such a pairing via adjoints. Additionally, we shall always consider the vacuum state normalized, $\langle\Omega, \Omega\rangle = 1$.

For the purpose of this paper, we shall assume that only one element of $\mathcal{K}$, say ρ_0, acts on Ω as a nonzero constant τ, so that the group element specified by equation (3) acts on Ω as follows:

$$\mathrm{e}^X\Omega = \mathrm{e}^{\tau u}\,\exp(v_j R_j)\Omega. \tag{5}$$

The system possesses two types of lowering and raising operators. The *algebraic lowering* and *raising* operators are defined simply by concatenation within the enveloping algebra (operator algebra generated by the representation) of $\mathfrak{g}$ followed by acting on Ω; that is,

$$\hat{R}_j\psi = R_j\psi$$
$$\hat{L}_j\psi = L_j\psi$$

for any linear combination ψ of basis elements (4). The 'hat' can be thus omitted without causing confusion. We shall also introduce *combinatorial raising operators*, $\mathcal{R}_j$, and *combinatorial lowering operators*, $\mathcal{V}_j$, acting on the basis as follows:

$$\mathcal{R}_j\ |k_1, k_2, \ldots, k_n\rangle = |k_1, k_2, \ldots, k_j + 1, \ldots, k_n\rangle$$
$$\mathcal{V}_j\ |k_1, k_2, \ldots, k_n\rangle\ = k_j\ |k_1, k_2, \ldots, k_j - 1, \ldots, k_n\rangle.$$

Notice that the operator $\mathcal{V}_j$ acts formally as the operator of partial differentiation with respect to the corresponding variable R_j.

The algebraic raising operators are represented directly by the $\mathcal{R}$'s, namely $\hat{R}_j = \mathcal{R}_j$. But the combinatorial lowering operators do not necessarily correspond to elements of $\mathfrak{g}$. The idea will be to express the *algebraic lowering operators* $\hat{L}_j$ (and hence the basis for $\mathfrak{g}$) also in terms of the operators $\{\mathcal{R}_j, \mathcal{V}_j\}$.

Let us introduce the coherent states as the image of the subgroup generated by the (abelian) subalgebra $\mathcal{R} \subset \mathfrak{g}$ in the Hilbert space $\mathcal{H}$ constructed above, namely

$$\psi_v = \exp(v_j R_j)\Omega.$$

Thus, the coherent states are parametrized by the elements $v_j R_j$ of $\mathcal{R}$, or, equivalently, by coordinates $v = (v_1, \ldots, v_n)$. We shall denote the manifold of coherent states as $\mathcal{C}$ with the parametrization $\mathcal{R} \to C$.

Observation 2.2. When restricted to coherent states, $\mathcal{R}_j$ acts as differentiation $\partial/\partial v_j$, while $\mathcal{V}_j$ acts as multiplication by v_j. Hence, we can determine the action of any operator defined as a (formal) operator function $f(\mathcal{R}, \mathcal{V})$, with all $\mathcal{V}$'s to the right of any $\mathcal{R}_j$, by (1) moving all $\mathcal{R}$'s to the right of all $\mathcal{V}$'s in the formula f, yielding the operator $\check{f}(\mathcal{V}, \mathcal{R})$, and then (2) replacing $\mathcal{V}_j \to v_j$ and $\mathcal{R}_j \to \partial/\partial v_j$, $1 \leqslant j \leqslant n$. Note that this is a formal Fourier transform combined with the *Wick ordering*. The Berezin transform extends this by taking the inner product with a coherent state ψ_w.

The following notion is very useful.

Definition 2.3. The *Leibniz function* is a map $\mathcal{C} \times \mathcal{C} \to \boldsymbol{C}$ defined as the inner product of the coherent states:

$$\Upsilon_{wv} = \langle \psi_w, \psi_v \rangle$$

for any v, w parametrizing $\mathcal{C}$.

Definition 2.4. The *coherent state representation* (the *Berezin transform*) is defined for an operator Q by

$$\langle Q \rangle_{wv} = \frac{\langle \psi_w, Q\psi_v \rangle}{\langle \psi_w, \psi_v \rangle}\,.$$

The algebraic raising operators can be expressed in terms of derivatives of the Leibniz function:

$$\langle \hat{R}_j \rangle_{wv} = \Upsilon^{-1} \frac{\partial}{\partial v_j} \Upsilon = \partial(\log \Upsilon)/\partial v_j$$
$$\langle \hat{L}_j \rangle_{wv} = \Upsilon^{-1} \frac{\partial}{\partial w_j} \Upsilon = \partial(\log \Upsilon)/\partial w_j$$

since L_j is adjoint to R_j. The right hand sides are functions in v and w. If one can eliminate w to reduce the above to a system of first-order partial differential equations

$$\frac{\partial \Upsilon}{\partial w_j} = \check{f}_j \left(v, \frac{\partial}{\partial v} \right) \Upsilon$$

for some operator functions $\check{f}_j$, then it is readily seen that

$$\hat{L}_j = f_j(\mathcal{R}, \mathcal{V})$$

provides a realization of the L's in terms of the combinatorial operators $\mathcal{R}$'s and $\mathcal{V}$'s. This complements the direct representation of the raising operators $\hat{R}_j = \mathcal{R}_j$. Note that the converse holds as well. Namely, if we have $\hat{L}_j$ expressed via $\mathcal{R}$ and $\mathcal{V}$, then Υ satisfies the corresponding partial differential equation. In some cases, this can be used to find Υ. (See section 4 below to see this illustrated in the present context.)

The final step is to find in our representation n commuting, self-adjoint operators X_j. They will generate a unitary group, $\exp(\mathrm{i} \sum_j s_j X_j)$, with $s = (s_1, \ldots, s_n) \in \boldsymbol{R}^n$ and $\mathrm{i} = \sqrt{-1}$. The scalar function defined by

$$\phi(s) = \left\langle \Omega, \exp\left(\mathrm{i} \sum_j s_j X_j \right) \Omega \right\rangle$$

will be required to be positive-definite. Then *Bochner's Theorem* assures that $\phi(s)$ is the Fourier transform of a positive measure, which gives the joint spectral density of the observables $(X_1, \ldots, X_n)$.

For so$(n, 2)$, we will identify these as a (multivariate) random variable on the Lorentz cone $\{x_1 > 0, x_1^2 > x_2^2 + \cdots + x_n^2\}$ in Minkowski space $\boldsymbol{R}^{n-1,1}$.

There are several ways to proceed with the outlined plan. One way is to study a matrix realization of the Lie algebra. Another is to start from the Leibniz function, which has been calculated in [7] and in [15], and reconstruct the Lie algebra from it. We will show how both of these approaches work.

3. Matrix version of so(n,2)

Consider the 'Lorentz' group SO(n,2) of transformations of the $(n + 2)$-dimensional real 'Minkowski' space $\boldsymbol{R}^{n,2}$ of signature $(n,2)$. The two-dimensional 'time' leads to two sets of independent boosts. Besides the spatial rotations, the group contains a one-dimensional subgroup, so(2), of temporal rotations. We shall start with the $(n+2) \times (n+2)$ skew-symmetric matrices $\rho_{kl} = E_{kl} - E_{lk}$, for $1 \leqslant k, l \leqslant n + 2$, where E_{ij} denotes the matrix consisting of zeros except for 1 at the (ij)-entry.

Notation In the following, indices j and k run from 1 to n, referring to n 'spatial coordinates.' Subscripts $n + 1$, $n + 2$ refer to 'time coordinates.'

Now define R's, L's and ρ_0:

Proposition 3.1. *The operators R_j, L_j, for $1 \leqslant j \leqslant n$, and ρ_0 defined by*

$$R_j = \rho_{j,n+2} + i\, \rho_{j,n+1} \qquad L_j = \rho_{j,n+2} - i\, \rho_{j,n+1} \qquad \rho_0 = 2i\, \rho_{n+1,n+2}$$

along with $\{\rho_{jk}\}$, $1 \leqslant j < k \leqslant n$, form a basis of so($n$,2) *corresponding to a Cartan decomposition as in equations (1) and (2).*

The following relations hold, the root space relations:

$$[\rho_0, R_j] = 2R_j \qquad [L_j, \rho_0] = 2L_j \qquad [\rho_{jk}, \rho_0] = 0$$

and

$$[L_j, R_j] = \rho_0 \qquad [L_k, R_j] = 2\rho_{jk} \qquad [\rho_{jk}, L_k] = L_j. \tag{6}$$

The involution (adjoint map) given by $R_j^* = L_j$ is effectively a complex conjugation. The commutation relations determine the involution for the remaining elements of $\mathfrak{g}$ (since $\mathcal{L}$ and $\mathcal{R}$ generate $\mathfrak{g}$ as a Lie algebra). Hence, ρ_0 is automatically symmetric, $\rho_0^* = \rho_0$, since it equals a commutator of mutually adjoint elements. Also, the ρ_{jk} are skew-symmetric with respect to this involution, as follows from relations $2\rho_{jk} = [L_k, R_j]$.

Now, we want to find commuting symmetric operators that will provide n commuting self-adjoint operators spanning $\mathfrak{g}$. (Note that even though we have complex numbers in the matrices, we in fact are using a 'real form' of $\mathfrak{g}$, admitting only real coefficients.) It turns out that conjugating by $\exp(L_1)$ almost 'does the job.' More precisely:

Proposition 3.2. *The elements*

$$\begin{aligned} X_1 &= 2(\rho_{1,n+2} + \mathrm{i}\, \rho_{n+1,n+2}) = R_1 + L_1 + \rho_0 \\ X_j &= 2\mathrm{i}\, (\rho_{1,j} + \mathrm{i}\, \rho_{n+1,j}) = -\mathrm{i}\, (R_j - L_j - 2\rho_{1j}) \end{aligned}$$

for $2 \leqslant j \leqslant n$, form a commuting family of Hermitian-symmetric elements in $\mathfrak{g}$.

Proof. Since the subalgebra $\mathcal{R}$ is abelian, conjugating it by a fixed element of the group will yield an abelian algebra. Calculating the adjoint group action

$$\exp(L_1) R_j \exp(-L_1) = R_j + [L_1, R_j] + \tfrac{1}{2}\, [L_1, [L_1, R_j]]$$

the commutation relations produce the indicated operators. (Note that in this exponential expansion, commutators beyond 2^{nd} order vanish.) For $j > 1$, the result is skew-symmetric, thus requiring the factor of $-\mathrm{i}$ for those X_j. □

Here are some explicit matrices for $n = 3$:

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mathrm{i} \\ -2 & 0 & 0 & -2\mathrm{i} & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 2\mathrm{i} & 0 & 0 & 0 \\ -2\mathrm{i} & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_3 = \begin{pmatrix} 0 & 0 & 2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2\mathrm{i} & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 3.3. Note that we *are* using a Hermitian structure here for the inner product so that multiplication by i converts a skew operator to a symmetric one.

3.1. Representation space

In the matrix formulation given above, we have a vacuum vector

$$\Omega = \begin{pmatrix} \bar{0} \\ 1 \\ \mathrm{i} \end{pmatrix}$$

where $\bar{0}$ stands for a column of n 0's. This vector satisfies

$$L_j\Omega = 0 \qquad \rho_{jk}\Omega = 0 \qquad \rho_0\Omega = -2\Omega.$$

Calculating with matrices, the coherent states can be found readily,

$$\exp(v_j R_j)\Omega = \begin{pmatrix} 2\mathrm{i}\,\boldsymbol{v} \\ 1 + v^2 \\ \mathrm{i}\,(1 - v^2) \end{pmatrix}$$

where $\boldsymbol{v}$ is a column vector with components v_j and $v^2 = v_j v_j$.

Recalling equations (3) and (5), this leads to:

Proposition 3.4. *Let g denote a group element, as in equation (5), then we can recover $\boldsymbol{v}$ and u from $g\Omega = \begin{pmatrix} \boldsymbol{v}_0 \\ a \\ b \end{pmatrix}$ by*

$$\boldsymbol{v} = \frac{1}{b + \mathrm{i}a}\,\boldsymbol{v}_0 \qquad \mathrm{e}^{-2u} = -\frac{1}{2}\,\frac{\boldsymbol{v}_0{}^{\top}\boldsymbol{v}_0}{a + \mathrm{i}b}$$

where $\top$ denotes transpose.

3.2. Leibniz function

Here we calculate the Leibniz function from the matrix representation. Since R's are adjoint to L's, we have

$$\Upsilon_{wv} = \langle\, \mathrm{e}^{w\cdot R}\Omega,\ \mathrm{e}^{v\cdot R}\Omega\,\rangle = \langle\,\Omega,\ \mathrm{e}^{w\cdot L}\mathrm{e}^{v\cdot R}\Omega\,\rangle$$

where, e.g., $\boldsymbol{w}\cdot L = w_j L_j$, and similarly for $\boldsymbol{v}\cdot R$. As a result we obtain:

Theorem 3.5. *In the matrix realization of so(n,2) given in proposition 3.1, the Leibniz function is*

$$\Upsilon_{wv} = 1 - 2\boldsymbol{w}^\top \boldsymbol{v} + w^2 v^2.$$

Proof. First compute (by matrix calculations)[1]

$$\exp(w_j L_j)\exp(v_j R_j)\Omega = \begin{pmatrix} 2\mathrm{i}(\boldsymbol{v} - v^2\boldsymbol{w}) \\ -2\boldsymbol{w}^\top\boldsymbol{v} + 1 + v^2 + w^2v^2 \\ -2\mathrm{i}\boldsymbol{w}^\top\boldsymbol{v} + \mathrm{i}\,(1 - v^2 + w^2v^2) \end{pmatrix}. \tag{7}$$

When this is expressed in factored form (cf. equations (3) and (5)), taking inner products with Ω eliminates all factors except for $\langle\Omega, \mathrm{e}^{\rho_0 u}\Omega\rangle$. In general, this is $\mathrm{e}^{\tau u}$ with u a function of v's and w's. In the matrix realization above, applying ρ_0 to Ω shows that $\tau = -2$. Now combine equation (7) with the result for $\exp(-2u)$ in proposition 3.4 to find the Leibniz function as stated. □

Generally, we want ρ_0 to act on Ω as multiplication by τ. This suggests that for $\mathrm{e}^{\tau u} = (\mathrm{e}^{-2u})^{-\tau/2}$ we have in general $\Upsilon_{wv} = (1 - 2\boldsymbol{w}^\top\boldsymbol{v} + w^2v^2)^{-\tau/2}$. We can now check agreement with the results in [7, 22], cf. the Bergman kernel function given in [15]. A main feature of the Leibniz function is that expanded in powers of v's and w's it yields the generating function for the inner products of elements of the basis for the Hilbert space. In general, there are conditions on the values of τ, the Gindikin set, for which the Hilbert space has a positive-definite inner product. In this regard, in addition to Berezin's paper, see [11].

3.3. Distribution of the observables

In the matrix representation, the raising operators are nilpotent, $R_j^3 = 0$, and hence $X_j^3 = 0$ for all $1 \leqslant j \leqslant n$. Consequently, the exponentials reduce to quadratics and the computations are very fast. With the vacuum vector as above, we find

$$\exp(z_j X_j)\Omega = \begin{pmatrix} 2\mathrm{i}(z_1 - \zeta^2) \\ 2z_2 \\ \vdots \\ 2z_n \\ 1 - 2z_1 + 2\zeta^2 \\ \mathrm{i}(1 - 2z_1) \end{pmatrix}$$

where $\zeta^2 = z_1^2 - \sum_{j\geqslant 2} z_j^2$. Applying proposition 3.4, we have:

[1] Matrix computations have been done using Maple V.

Proposition 3.6. *Let* $h^2 = (1-z_1)^2 - \sum_{j\geqslant 2} z_j^2$. *For a group element generated by the* X_j *acting on the vacuum,* $\exp(z_j X_j)\Omega$, *the* v *and* u *variables are given according to*

$$\boldsymbol{v} = \frac{1}{h^2}\begin{pmatrix} 1 - z_1 - h^2 \\ -iz_2 \\ \vdots \\ -iz_n \end{pmatrix} \qquad \exp(-2u) = h^2.$$

With $e^{\tau u} = h^{-\tau} = \left((1-z_1)^2 - z_2^2 - \cdots - z_n^2\right)^{-\tau/2}$ as the Fourier–Laplace transform of the joint spectral density of the X_j, we can identify it as a measure on the Minkowski cone $\{x_1 > 0, x_1^2 > x_2^2 + \cdots + x_n^2\}$ in $\boldsymbol{R}^{n-1,1}$. See, e.g., [19] as well as the references mentioned above for determining positivity. Up to an exponential factor in x_1 the density is the *Wishart distribution on the Lorentz cone.* (see [11, ch XVI] and Casalis [9]. The important feature is that the positivity implies (means) that we have the Fourier–Laplace transform of probability measures which are given by a function raised to a power in the Fourier domain. Thus, the measures form a convolution family and with a continuous parameter $\tau = t/\hbar$, we have the fundamental solution to an evolution equation with generator $u(D)$, replacing $(z_1, \ldots, z_n)$ in u as a function of z by $(D_1, \ldots, D_n)$, $D_j = \mathrm{d}/\mathrm{d}x_j$. Thus, $u(D)$ is a 'natural Hamiltonian' (generator of time-translations) associated to the Lie algebra.

Now, we have two expressions, hence coordinate systems, for a coherent state,

$$\exp(z_j X_j)\Omega = e^{\tau u}\exp(v_j R_j)\Omega \tag{8}$$

where $u = u(z)$ and $v_j = v_j(z)$ are functions of $z = (z_1, \ldots, z_n)$. In order to express the basis of the Hilbert space in terms of the X's rather than R's, we must solve (8) for the z_j in terms of the v's. This will give the *generating function* for the basis expressed in terms of the X_j, written in spectral form as variables x_j.

In general,

$$\begin{aligned} e^{v_j R_j}\Omega &= \exp\left[x_j z_j(v) - \tau\, u(z(v))\right] \\ &= \sum_{k_1,\ldots,k_n} \frac{v_1^{k_1}\cdots v_n^{k_n}}{k_1!\cdots k_n!}\,|k_1, \ldots, k_n\rangle \end{aligned}$$

where $z_j = z_j(v)$ are the components of the (functional) inverse to $v(z)$.

Theorem 3.7. *The generating function for the basis* $|k_1, \ldots, k_n\rangle$ *is*

$$\exp\left(\frac{x_1(v_1+v^2) + \mathrm{i}\,(\boldsymbol{x}\cdot\boldsymbol{v} - x_1 v_1)}{1 + 2v_1 + v^2}\right)(1 + 2v_1 + v^2)^{-\tau/2}$$

where $v^2 = v_j v_j$ *and* $\boldsymbol{x}\cdot\boldsymbol{v} = x_j v_j$.

Proof. To start, note that $e^{-2u} = h^2$ entails $e^{-\tau u} = h^{\tau/2}$. Now we must solve for the v's in proposition 3.6. First,

$$v_1 = (1 - z_1 - h^2)/h^2 \qquad \text{implies} \qquad 1 - z_1 = h^2(1 + v_1).$$

Next, with $z_k = \mathrm{i}h^2 v_k$, we square and re-sum on the left-hand side to yield

$$h^2 = h^4\left((1+v_1)^2 + \sum_{k\geqslant 2} v_k^2\right)$$

from which $h^2 = (1 + 2v_1 + v^2)^{-1}$. Expressing h^2 in terms of v's in the above expressions for the z_j yields the result. □

For $n = 1$, the terms for $k \geqslant 2$ drop out, reducing to a generating function for Laguerre polynomials. Hence, in general, the basis in the x-variables offers a generalization of the classical Laguerre polynomials.

4. Leibniz function and the Lie algebra

Now we shall show how the Lie algebra structure expressed in terms of the combinatorial raising and lowering operators $\mathcal{R}_j$ and $\mathcal{V}_j$, the hat-representation, can be constructed from the Leibniz function. (Recall remark 2.2.)

Theorem 4.1. *The hat-representation has the form*

$$\hat{R}_j = \mathcal{R}_j \qquad \hat{L}_j = \tau\mathcal{V}_j + 2\,(\mathcal{R}_l\mathcal{V}_l)\,\mathcal{V}_j - \mathcal{R}_j\mathcal{V}^2$$

for the algebraic raising and lowering operators, while the rotation operators are given by

$$\hat{\rho}_0 = \tau + 2\mathcal{R}_l\mathcal{V}_l \qquad \hat{\rho}_{jk} = \mathcal{R}_j\mathcal{V}_k - \mathcal{R}_k\mathcal{V}_j$$

Proof. Theorem 3.5 provides us the Leibniz function for our representation of so(n,2)

$$\Upsilon = (1 - 2\boldsymbol{w}^\top\boldsymbol{v} + w^2v^2)^{-\tau/2}.$$

Differentiating, we obtain

$$\frac{1}{\Upsilon}\,\frac{\partial\Upsilon}{\partial w_j} = \tau\,\frac{v_j - w_jv^2}{1 - 2\boldsymbol{w}^\top\boldsymbol{v} + w^2v^2}$$

and

$$\frac{1}{\Upsilon}\,\frac{\partial\Upsilon}{\partial v_j} = \tau\,\frac{w_j - v_jw^2}{1 - 2\boldsymbol{w}^\top\boldsymbol{v} + w^2v^2}.$$

Combining these, we find the system of partial differential equations

$$\frac{\partial\Upsilon}{\partial w_j} = \tau\,v_j\Upsilon + 2v_j\frac{\partial\Upsilon}{\partial v_l}v_l - \frac{\partial\Upsilon}{\partial v_j}\,v^2.$$

(implied summation over l in the middle term) from which we can read off $\hat{L}_j$. Since $\hat{R}_j = \mathcal{R}_j$, taking the commutator with $\hat{L}_j$, the first relation in equation (6) yields $\hat{\rho}_0 = \tau + 2\mathcal{R}_l\mathcal{V}_l$. And from the middle relation in equation (6),

$$[\hat{L}_k, \hat{R}_j] = 2(\mathcal{R}_j\mathcal{V}_k - \mathcal{R}_k\mathcal{V}_j)$$

yields $\hat{\rho}_{jk}$. □

One can easily check the adjoint action of $\hat{\rho}_0$ as well as the remaining commutation relations corresponding to equation (6). Finally, note that $\hat{L}_j$ can be written in the form

$$\hat{L}_j = \hat{\rho}_0\mathcal{V}_j - \mathcal{R}_j\mathcal{V}^2$$

which is a variation on the Bessel operator.

5. Conclusion

An important feature of our approach is the identification of an interesting abelian subalgebra that provides a family of commuting observables from the viewpoint of quantization. The special element ρ_0 turns out to be dual to a natural Hamiltonian generating a convolution semigroup of measures yielding the joint spectral density of the observables of interest. It is important to note that in Berezin, e.g., [7], what is considered as Planck's constant should be in fact the ratio $t/\hbar$, namely the ratio between a time variable and a fixed constant. In this paper, the time in so(n,2) is represented by (imaginary) so(2) and in the representation space by a positive real variable, the corresponding weight. The physical interpretation of this aspect remains to be explored.

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